Math 3063 Abstract Algebra — Prof. Paul Bailey — Project 1 — January 30, 2007 — Name:

Complete these problems and turn in the solution by Monday, February 5, 2007. Attach this page to the front of the solutions. Solutions should be self explanatory and written in complete sentences.

Permutations of a Set of Cardinality 3 or 4

A *permutation* on a set X is a bijective function $X \to X$. The set of all permutations of X is denoted $\mathrm{Sym}(X)$. We use function composition to combine elements of $\mathrm{Sym}(X)$, and in this context, we may refer to function composition as multiplication, and to the composition of two permutations as their product.

We use greek letters to denote the elements of $\mathrm{Sym}(X)$. The identity permutation is denoted ϵ.

Let $\alpha \in \mathrm{Sym}(X)$ and $x \in X$. We say that x is *fixed* by α, or is a *fixed point* of α, if $\alpha(x) = x$. The *support* of α is the set of nonfixed points. If $\alpha, \beta \in \mathrm{Sym}(X)$, we say that α and β are *disjoint* if the intersection of their supports is empty.

If $X = \{1, \dots, n\}$, we set $S_n = \mathrm{Sym}(X)$. The cardinality of S_n is $|S_n| = n!$.

Let $i_1, \dots, i_k \in S_n$. Then *cycle* $\alpha = (i_1\ i_2\ \dots\ i_k)$ is the permutation defined by

$$\alpha(n) = \begin{cases} i_{j+1} & \text{if } n = i_j \text{ and } j \in \{1, \dots, k-1\}; \\ i_1 & \text{if } n = i_k; \\ n & \text{if } n \notin \{i_1, \dots, i_k\}. \end{cases}$$

Every element of S_n may be written as the product of disjoint cycles.

For example:

- $S_1 = \{\epsilon\}$
- $S_2 = \{\epsilon, (1\ 2)\}$
- $S_3 = \{\epsilon, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$

Problem 1. Let $\rho = (1\ 2\ 3)$ and $\tau = (1\ 2)$. Write every element of S_3 in terms of ϵ, ρ, and τ. Using this representation, construct a multiplication table for S_3.

The *shape* of a permutation is the sorted list consisting of the sizes of its nontrivial disjoint cycles. We put brackets around such a list. For example:

- $\mathrm{shape}(\epsilon) = []$
- $\mathrm{shape}(1\ 2\ 3) = [3]$
- $\mathrm{shape}(1\ 2)(3\ 4) = [2, 2]$
- $\mathrm{shape}(1\ 2)(3\ 4\ 5\ 9\ 10)(6\ 7\ 8) = [2, 3, 5]$

Problem 2. Find all 24 elements of S_4, and classify them by their shape. Find how many elements there are in S_4 of each shape.

Problem 3. Let $\rho, \tau \in S_4$ be given by $\rho = (1\ 2\ 3\ 4)$ and $\tau = (1\ 3)$. The set $D_4 = \{\epsilon, \rho, \rho^2, \rho^3, \tau, \tau\rho, \tau\rho^2, \tau\rho^3\}$ is closed under multiplication. Construct its multiplication table.